Math 210C Lecture 18 Notes

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1 Derived Functors and Injective Resolutions

1.1 Left derived functors

Let $\mathcal{C}$ be an abelian category. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor of abelian categories and let $f : P \to Q$ in $\mathcal{C}$. Then $F(f_*) : H_i(F(P)) \to H_i(F(Q))$ is "well-defined" in $\mathcal{D}$.

Proposition 1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. For each $A \in \mathcal{C}$, choose a projective resolution $P_\cdot^A \to A$. For each $A : A\ toB$ choose $f_\cdot : P_\cdot^A \to P_\cdot^B$ augmenting g. Then $L_iF : \mathcal{C} \to \mathcal{D}$ given by $L_iF(A) = H_i(F(P_\cdot^A))$ and $L_iF(g) = F(f_*)$ is a functor (unique up to unique isomorphism).*

Definition 1.1. L_iF is called the **i-th left derived functor** of F.

Proof. This mostly follows from a proposition from last time. Check the remaining properties, such as composition. $\square$

Lemma 1.1. *If F is right exact, then $L_0F \cong F$.*

Proof. Let $P_0 \to A \in \mathcal{C}$ in a projective resolution of A. Then $L_iF(A) = H_iF(P_\cdot)$, so $L_0F(A) = H_0(F(P_\cdot))$, so

$$P_1 \longrightarrow P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

gives

$$F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{\varepsilon} F(A) \longrightarrow 0$$

Then

$$\begin{tikzcd} H_0(F(P_\cdot)) = \operatorname{coker}(F(d_1)) \arrow[r, "\cong"] \arrow[d, "F(f)_*"] & F(A) \arrow[d, "F(g)"] \\ H_0(F(Q_\cdot)) \arrow[r, "\cong"] & F(B) \end{tikzcd}$$

and the left hand side of this diagram is actually

$$\begin{tikzcd} L_0F(A) \arrow[d, "L_0F(g)"] \\ L_0F(B) \end{tikzcd}$$

□

Theorem 1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be right exact. For each short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{C}$, there exists a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & L_2F(C) \arrow[r, "\delta"] & L_1F(A) \arrow[r] & L_1F(B) \arrow[r] & L_1F(C) \arrow[dlll, "\delta"'] \\ & F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r] & 0 \end{tikzcd}$$

which is natural in the short exact sequence (morphism of δ-functors): if

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

then

$$\begin{tikzcd} L_{i+1}F(C) \arrow[r, "\delta"] \arrow[d] & L_iF(A) \arrow[d] \\ L_{i+1}F(C') \arrow[r, "\delta"] & L_{i+1}F(A') \end{tikzcd}$$

Remark 1.1. This functor $(L_iF, \delta_i)_i$ satisfies a universal property and is called a **δ-functor**.

Proof. If we have a short exact sequence of complexes

$$0 \longrightarrow P^A_\cdot \longrightarrow P^B_\cdot \longrightarrow P^C_\cdot \longrightarrow 0$$

this stays exact since $P^B_i = P^A_i \oplus P^C_i$ and F is additive:

$$0 \longrightarrow F(P^A_\cdot) \longrightarrow F(P^B_\cdot) \longrightarrow F(P^C_\cdot) \longrightarrow 0$$

So we get a long exact sequence on cohomology

$$\cdots \longrightarrow H_{i+1}(F(P^C_\cdot)) \xrightarrow{\delta} H_i(F(P^A_\cdot)) \longrightarrow H_i(F(P^B_\cdot)) \longrightarrow H_i(F(P^C_\cdot)) \longrightarrow \cdots$$

This is

$$\cdots \longrightarrow L_{i+1}F(C) \xrightarrow{\delta} L_iF(A) \longrightarrow L_iF(B) \longrightarrow L_iF(C) \longrightarrow \cdots$$

□

1.2 Injective resolutions and right derived functors

Definition 1.2. A **(comological) resolution** of an object A is an augmented complex

$$0 \longrightarrow A \longrightarrow C_{\cdot}$$

which is exact. It is **injective** if the objects of C are injective. $\mathcal{C}$ **has enough injectives** if for all $A \in \mathcal{C}$, there is an injective $I \in \mathcal{C}$ and a monomorphism $A \to I$.

Proof. If $\mathcal{C}$ is abelian, then $\mathcal{C}$ has enough injectives if and only if every object has an injective resolution. □

Proposition 1.2. *R-Mod has enough injectives.*

Proof. If $R = \mathbb{Z}$, let A be an abelian group. Then $A \cong \bigoplus_{j \in J} /T$, where $T \leq \bigoplus j \in J\mathbb{Z}$, where J is an index set. This maps into $\bigoplus_{j \in J} \mathbb{Q}/T$, which is divisible and hence injective.

If R is arbitrary, let A be an R-module. Let $\phi : \mathrm{Hom}_{\mathbb{Z}}(R, A)$, where $|phi(a)(r) = ra$ is a map of R-modules. As abelian groups, A injects into a divisible abelian group D. So $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ injects into $\mathrm{Hom}_{\mathbb{Z}}(R, D)$. From homework, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective as an R-module. □

Definition 1.3. If $\mathcal{C}$ is an abelian category with enough injectives and $F : \mathcal{C} \to \mathcal{D}$ is left exact, then the ***i*-th right derived functor** $R^i F : \mathcal{C} \to \mathcal{D}$ is $P^i F(A) = H^i(F(I^{\cdot}))$ for $A \to I^{\cdot}$ an injective resolution. For $f : A \to B$ and $g : I^{\cdot} \to J^{\cdot}$ such that

$$\begin{array}{ccc} A & \longrightarrow & I^{\cdot} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \longrightarrow & J^{\cdot} \end{array}$$

we have $R^i F(f) = F(g)^*$.

Theorem 1.2. *The R^i give a cohomological δ-functor (universal):*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives a long exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \xrightarrow{\delta^0} R^1 F(A) \longrightarrow R^1 F(B) \longrightarrow \cdots$$

1.3 Tor functors

Lemma 1.2. *Let R be a ring, and let N be an R^{op}-module. Then we have the tensor product functor $t_N : R\text{-Mod} \to \mathrm{Ab}$ given by $t_N(M) = N \otimes_R M$ and $t_N(f) = \mathrm{id}_N \otimes f$. Then t_N is right exact.*

Proof. t_N is additive because direct sums commute with tensor products. If we have a right exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

then $N \otimes_R C$ is generated by the elements $n \times c$ with $c \in C$ and $n \in N$. If $c = g(b)$, then $t_N(g)(b) = n \otimes c$, so $t_N(g)$ is surjective. So we get a surjection $\operatorname{coker}(t_n(f)) \to N \otimes_R C$. We want this to be an isomorphism, so let's try to create an inverse. Define an R-balanced map $N \times C \to \operatorname{coker}(T_N(f))$ by $\theta(n,c) = n \otimes b + \operatorname{im}(t_n(f))$. Then $g(b) = c$. This is well-defined because if $g(b') = c$, then $g(b - b') = 0$, so $b - b' = f(a)$ for some a; then $n \otimes b - n \otimes b' + t_N(f)(a)$. So we've constructed an inverse to the surjection. □

Definition 1.4. The ***i*-th Tor functor** is $\operatorname{Tor}_i^R(N, \cdot) := L_i t_N$.